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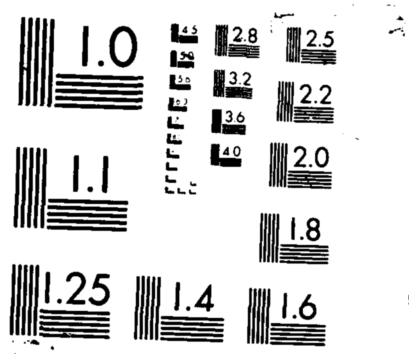

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where $\lambda_1,\ldots,\lambda_q$ are unknown complex parameters with module 1, $\lambda_{q+1},\ldots,\lambda_p$ are unknown complex parameters with module less than 1, $\lambda_1,\ldots,\lambda_p$ are assumed distinct, p assumed known and q unknown. a_{kj}, $k=1,\ldots,p$, $j = 1,\ldots,N$ are unknown complex parameters. $e_j(t)$, $t = 0,1,\ldots,n-1$, $j = 1,\ldots,N$, are i.i.d. complex random noise variables such that

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1. A strong consistent estimate of q;
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ABSTRACT

This paper studies ~~the~~ model of superimposed exponential signals in noise:

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1. INTRODUCTION

Consider the model

$$Y_j(t) = \sum_{k=1}^{p} a_{kj}\lambda_k^t + e_j(t), \tag{1.1}$$

$$t=0,1,\ldots,n-1, \quad j=1,2,\ldots,N$$

where $\lambda_1,\ldots,\lambda_p$ are unknown complex parameters with module not greater than one, and are assumed distinct from each other, a_{kj}, $k=1,2,\ldots,p$, $j=1,2,\ldots,N$, are unknown complex parameters, $e_j(t)$, $t=0,1,\ldots,n-1$, $j=1,\ldots,N$, are iid. complex random noise variables such that

$$Ee_1(0) = 0, \quad Ee(0)\overline{e_1(0)} = \sigma^2, \quad 0 < \sigma^2 < \infty, \tag{1.2}$$

$$E|e_1(0)|^4 < \infty, \tag{1.3}$$

where σ^2 is unknown. Throughout this paper, $i=\sqrt{-1}$, $\overline{A}$, A' and A^* denote the complex conjugate, the transpose and the complex conjugate of the transpose of a matrix A respectively.

The model (1.1) can be viewed either as an ordinary time series (single-experiment for N=1, multiple-experiment for N>1) with uniform sampling, or as a model for a linear uniform narrow-band array with multiple plane waves present, and each measurement vector (the "snapshort") $\underset{\sim}{Y}_j = (Y_j(0),\ldots,Y_j(n-1))'$ represents the output from n individual sensors.

The primary interest in this model is to estimate the parameter vector $\underset{\sim}{\lambda} = (\lambda_1,\ldots,\lambda_p)'$ based on the data $\{\underset{\sim}{Y}_j, j=1,\ldots,N\}$. In some investigations, for example [1], [2] and [3], it is assumed that the vector $\underset{\sim}{a}_j = (a_{1j}, a_{2j}, \ldots, a_{pj})'$, $j = 1,\ldots,N$, are iid. random vectors with a common mean vector zero and covariance matrix $R = E\underset{\sim}{a}_j\underset{\sim}{a}_j^*$. In other studies, for example [4], it is assumed that a_{kj}, the complex amplitude of the k-th signal in the j-th snapshort, is simply an unknown constant, and it is desired to estimate these constants based on the data. We shall adopt the latter assumption in this paper.

Various methods for estimating the parameters $\underset{\sim}{\lambda}$ and $\underset{\sim}{a}_j$'s are proposed in the literature. If $\underset{\sim}{\lambda}$ were known, the least squares (LS) method would give the following estimate of $\underset{\sim}{a}_j$:

$$\hat{\underset{\sim}{a}}_j = (A^*(\underset{\sim}{\lambda})A(\underset{\sim}{\lambda}))^{-1}A^*(\underset{\sim}{\lambda})\underset{\sim}{Y}_j, \quad j = 1,2,\ldots,N, \tag{1.4}$$

where

$$A(\underset{\sim}{\lambda}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_p^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_p^{n-1} \end{pmatrix}.$$

From this consideration, some authors suggested that, after obtaining some estimate $\hat{\underset{\sim}{\lambda}}$ of $\underset{\sim}{\lambda}$, one substitutes $\hat{\underset{\sim}{\lambda}}$ for $\underset{\sim}{\lambda}$ in (1.4) to yield an estimate of $\underset{\sim}{a}_j$. For estimation of $\underset{\sim}{\lambda}$, Bresler and Macovski [4] derived the maximum likelihood (ML) criterion under the normality assumption on $\{e_j(t)\}$, which is

just the LS criterion. Other methods are proposed such as that of Prony, Pisarenko and modifications thereof (e.g. [5], [6], [7], [8]). Not much is known about the statistical properties of these estimates. For some results in this respect, the reader is referred to Bai, Krishnaiah and Zhao [9]. They considered the case where N=1 and λ_k's are all of module one, suggested an equivariation linear prediction (EVLP) method to detect the number p of signals, and to estimate $\underset{\sim}{\lambda}$ and σ^2. They established the strong consistency of the detection criterion and estimators, and obtained the limiting distributions of related estimators. Analysis and comparison for some estimates of $\underset{\sim}{\lambda}$ are also made.

In this paper, we apply the EVLP method to the general model (1.1). This method is a modification of a classical method dating back to Prony [10]. As pointed out by Rao [11], the Prony method, which features in minimizing certain quadratic form of the observations, ignores the correlation of related linear forms therein, and the consistency of the related estimates is in doubt.

Roughly, the EVLP method can be described as follows: Consider the set

$$B_p = \{\underset{\sim}{b} = (b_0,\ldots,b_p)':\sum_{k=0}^{p}|b_k|^2 = 1,\ \ b_p \geq 0,\ \ b_0,\ldots,b_{p-1} \text{ complex}\}. \tag{1.5}$$

Define a function $Q_p(\underset{\sim}{b})$ as follows:

$$Q_p(\underset{\sim}{b}) = \frac{1}{N(n-p)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-p} \left| \sum_{k=0}^{p} b_k Y_j(t+k) \right|^2, \quad \underset{\sim}{b} \in B_p. \tag{1.6}$$

We can find a vector $\hat{\underset{\sim}{b}} \in B_p$ such that

$$Q_p \triangleq Q_p(\hat{\underset{\sim}{b}}) = \min\{Q_p(\underset{\sim}{b}):\underset{\sim}{b} \in B_p\} \tag{1.7}$$

Let $\hat{\lambda}_1,\ldots,\hat{\lambda}_p$ be the roots of the equation

$$\sum_{k=0}^{p} \hat{b}_k z^k = 0. \tag{1.8}$$

Then $\hat{\lambda}_k$ is taken as an estimate of λ_k, $k=1,\ldots,p$.

For considering the asymptotic properties of the estimates, we shall distinguish the following two cases:

Case (i): N is fixed and n tends to infinity;

Case (ii): n is fixed and N tends to infinity.

First, consider case (i). Put

$$\Lambda = \{k: |\lambda_k| = 1, \quad 1 \leq k \leq p\}, \quad \Lambda^c = \{k: \quad |\lambda_k| < 1, \quad 1 \leq k \leq p\}. \tag{1.9}$$

Without loss of generality, we can assume that

$$\Lambda = \{1,2,\ldots,q\} \text{ for some } \quad q \leq p. \tag{1.9)'$$

We shall show in the sequel that there will be no consistent estimate for λ_k, $k \in \Lambda^c$, when $\Lambda^c \neq \phi$ (c.f. Theorem 4.1), and if $\Lambda^c \neq \phi$, the above procedure fails to provide consistent estimates for $\lambda_1,\ldots,\lambda_q$ (c.f. Remark 3.1). In view of this, it is important to seek for a consistent estimate $\hat{q}$ of $q \triangleq \#(\Lambda)$. This enables us to use q to replace p in procedure (1.5)-(1.8) to obtain estimates of $\lambda_1,\ldots,\lambda_q$.

Having obtained estimate $\hat{\underset{\sim}{\lambda}}$ of $\underset{\sim}{\lambda}$, estimates of a_j's can be obtained by replacing $\underset{\sim}{\lambda}$ by $\hat{\underset{\sim}{\lambda}}$, as mentioned earlier. At a first look it would suggest that the estimates of $\underset{\sim}{a}_j$'s so obtained should be consistent when $\hat{\underset{\sim}{\lambda}}$ is a consistent estimate of $\underset{\sim}{\lambda}$. In fact, this is not true. The reason is that in order to get consistent estimates of $\underset{\sim}{a}_j$'s by this method, $\hat{\lambda}_k^r - \lambda_k^r$ should

be of the order $o_p(1)$ for $r \leq n - 1$. But usually $\hat{\lambda}_k - \lambda_k$ is only of the order $O_p(1/\sqrt{n})$, and $\hat{\lambda}_k^{n-1} - \lambda_k^{n-1}$ cannot have the order $o_p(1)$. However, it is possible to estimate $|a_{kj}|$ consistently, where $k=1,2,\ldots,q$, $j=1,2,\ldots,N$.

In section 2, we give a detection procedure for q, and give some estimates of λ_k, σ^2 and $|a_{kj}|$ for $k=1,\ldots,q$ and $j=1,2,\ldots,N$. In section 3, we establish the strong consistency of these procedures, and find the limiting distributions for some estimates. In section 4, we show the non-existence of consistent estimates for λ_k and a_{kj}, where $k=q+1,\ldots,p$, $j=1,\ldots,N$. Finally, section 5 is devoted to a brief discussion of case (ii).

The strong consistency of the LS estimation of λ_k, $k \in \Lambda$, shall be established in a forth-coming paper [12].

2. DETECTION AND ESTIMATION PROCEDURES

In this section, it is desired to determine $q=\#(\Lambda)$, and to estimate λ_k, σ^2 and $|a_{kj}|$, $k=1,2,\ldots,q$, $j=1,2,\ldots,N$ (refer to (1.9) and (1.9)'). Throughout this section, N is fixed and n tends to infinity, and the following conditions are assumed:

$$\lambda_k \neq 1, \quad k=1,2,\ldots,q; \qquad \lambda_k \neq \lambda_\ell \quad \text{for } k\neq\ell, k,\ell=1,\ldots,q, \tag{2.1}$$

and

$$\sum_{j=1}^{N} |a_{kj}| > 0 \quad \text{for} \quad k=1,2,\ldots,q. \tag{2.2}$$

For detection problem, we also assume that (1.2) and (1.3) are satisfied. For $r=0,1,2,\ldots,p$, define a set of complex vectors

$$B_r = \left\{ \underset{\sim}{b}^{(r)} = (b_0^{(r)},\ldots,b_r^{(r)})' : b_r^{(r)} \geq 0, \text{ and } \sum_{k=0}^{r} |b_k^{(r)}|^2 = 1 \right\} \tag{2.3}$$

and a quadric form of $\underset{\sim}{b}^{(r)}$:

$$Q_r(\underset{\sim}{b}^{(r)}) = \frac{1}{N(n-r)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-r} \left| \sum_{k=0}^{r} b_k^{(r)} Y_j(t+k) \right|^2, \quad \underset{\sim}{b}^{(r)} \in B_r. \tag{2.4}$$

Put

$$Q_r = \min \left\{ Q_r(\underset{\sim}{b}^{(r)}), \quad \underset{\sim}{b}^{(r)} \in B_r \right\}. \tag{2.5}$$

Choose constant C_n satisfying the following conditions:

$$\lim_{n\to\infty} C_n = 0, \quad \lim_{n\to\infty} \sqrt{n}\, C_n / \sqrt{\log \log n} = \infty. \tag{2.6}$$

Then we find the nonnegative integer $\hat{q} \leq p$ minimizing

$$I(r) = Q_r + rC_n, \quad r=0,1,\ldots,p, \tag{2.7}$$

and use $\hat{q}$ as an estimate of q.

Note that if $\hat{\underset{\sim}{b}}^{(r)} = (\hat{b}_0^{(r)},\ldots,\hat{b}_r^{(r)})'$ satisfies

$$Q_r = \frac{1}{N(n-r)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-r} \left| \sum_{k=0}^{r} \hat{b}_k^{(r)} Y_j(t+k) \right|^2 ,$$

then Q_r is the smallest eigenvalue of the matrix

$$\hat{\Gamma}^{(r)} = (\hat{\gamma}_{\ell m}^{(r)}), \quad \ell,m=0,1,\ldots,r,$$

and $\hat{\underset{\sim}{b}}^{(r)}$ is the corresponding eigenvector, where

$$\hat{\gamma}_{\ell m}^{(r)} = \frac{1}{N(n-r)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-r} \overline{Y_j(t+\ell)}\, Y_j(t+m), \quad \ell,m=0,1,\ldots,r. \tag{2.8}$$

Put $\lambda_k = \exp(i\omega_k)$ for $k=1,\ldots,q$. As shown in section 3, with probability one, we have $\hat{q}=q$ for n large. Hence, to estimate $\omega_1,\ldots,\omega_q$, without loss of generality, we can assume that $q=\#(\Lambda)$ is known. For simplicity we write $\hat{\Gamma}^{(q)}=\hat{\Gamma}$, $\hat{\gamma}_{\ell m}^{(q)}=\hat{\gamma}_{\ell m}$, etc. Let $\hat{\underset{\sim}{b}}=(\hat{b}_0,\ldots,\hat{b}_q)' \in B_q$ be a eigenvector of $\hat{\Gamma}$ associated with its smallest eigenvalue. Under the conditions (1.2), (2.1) and (2.2), it can be shown that with probability one for n large, the equation

$$\hat{B}(z) \triangleq \sum_{k=0}^{q} \hat{b}_k z^k = 0 \tag{2.9}$$

has q roots, namely $\hat{\rho}_k \exp(i\hat{\omega}_k)$, $k=1,2,\ldots,q$, where $\hat{\rho}_k \geq 0$, $\hat{\omega}_k \in (0,2\pi)$, $k \leq q$.

Further, Q_q furnishes an estimate of σ^2.

To estimate $|a_{kj}|$, $k=1,\ldots,q$, $j=1,\ldots,N$, write $\hat{\lambda}_k = \exp(i\omega_k)$, $k=1,\ldots,q$, and write approximately (1.1) as

$$\begin{pmatrix} Y_j(t+0) \\ Y_j(t+1) \\ \vdots \\ Y_j(t+q-1) \end{pmatrix} \simeq \hat{\underset{\sim}{A}} \begin{pmatrix} a_{1j}\lambda_1^t \\ a_{2j}\lambda_2^t \\ \vdots \\ a_{qj}\lambda_q^t \end{pmatrix} + \begin{pmatrix} e_j(t+0) \\ e_j(t+1) \\ \vdots \\ e_j(t+q-1) \end{pmatrix} \tag{2.10}$$

$$t=0,1,\ldots,n-q, \quad j=1,\ldots,N,$$

where

$$\hat{A}_* = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \hat{\lambda}_1 & \hat{\lambda}_2 & \cdots & \hat{\lambda}_q \\ \vdots & \vdots & & \vdots \\ \hat{\lambda}_1^{q-1} & \hat{\lambda}_2^{q-1} & \cdots & \hat{\lambda}_q^{q-1} \end{pmatrix}.$$

Put $\hat{A}_*^{-1} = \hat{M} = (\hat{\mu}_{\ell m})$, $\ell,m=1,2,\ldots,q$. Motivated by (2.10), we propose the following estimate of $|a_{kj}|^2$, $k \leq q$, $j \leq N$:

$$|a_{kj}|^2 = \left(\frac{1}{n-q+1} \sum_{t=0}^{n-q} \sum_{\ell=1}^{q} \hat{\mu}_{k\ell} Y_j(t+\ell-1)|^2 - \sum_{\ell=1}^{q} |\hat{\mu}_{k\ell}|^2 Q_q \right)_+, \tag{2.11}$$

where for any real x,

$$(x)_+ = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Remark 2.1. If we consider the more general model

$$Y_j(t) = a_{0j} + \sum_{k=1}^{p} a_{kj} \lambda_k^t + e_j(t), \tag{2.12}$$

$$t=0,1,\ldots,n-1, \quad j=1,\ldots,N,$$

where a_{0j} is an unknown constant, $\lambda_k \neq 1$, $\lambda_k \neq \lambda_\ell$ for $k \neq \ell$, $k,\ell=1,\ldots,q$. We can use $\hat{a}_{0j} = \sum_{t=0}^{n-1} Y_j(t)/n$ to estimate a_{0j}. Then the above procedures of detection and estimation can be used with $Y_j(t)$ replaced by $Y_j(t) - \hat{a}_{0j}$.

3. ASYMPTOTIC BEHAVIOUR OF THE DETECTION AND ESTIMATION

In this section, we establish the strong consistency of the detection and estimation procedures proposed in section 2. The asymptotic normality of some estimates is also established. Throughout this section, N is fixed and n tends to infinity.

Some known results are needed in the following discussion. For convenience of reference, we state these as lemmas.

LEMMA 3.1. Let $\{X_n, n\geq 1\}$ be a sequence of independent real random variables such that $\sum_{n=1}^{\infty} E|X_n| < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges a.s.

Refer to Stout([13], 1974, p.94).

LEMMA 3.2.(Petrov). Let $\{X_n, n\geq 1\}$ be a sequence of independent real random variables with zero means. Write $B_n^2 = \sum_{j=1}^{n} EX_j^2$ and $S_n = \sum_{j=1}^{n} X_j$. If

$$\liminf_{n\to\infty} B_n^2/n > 0$$

and

$$E|X_j|^{2+\delta} \leq K < \infty, \quad j \geq 1$$

for some constants K and $\delta > 0$, then

$$\limsup_{n\to\infty} S_n/(2B_n^2 \log\log B_n^2)^{1/2} = 1 \quad \text{a.s.}$$

For a proof, the reader is referred to Petrove ([14], p.306) and Stout ([13], p.274).

LEMMA 3.3. Let $A = (a_{jk})$ and $B = (b_{jk})$ be two Hermitian p×p matrices with spectrum decompositions

$$A = \sum_{k=1}^{p} \delta_k \underset{\sim}{u}_k \underset{\sim}{u}_k^*, \quad \delta_1 \geq \delta_2 \geq \cdots \geq \delta_p$$

and

$$B = \sum_{k=1}^{p} \lambda_k \underset{\sim}{v}_k \underset{\sim}{v}_k^*, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p.$$

Further, we assume that

$$\lambda_{n_{h-1}+1} = \cdots = \lambda_{n_h} = \tilde{\lambda}_h, \quad n_0 = 0 < n_1 < \cdots < n_s = p,$$

$$\tilde{\lambda}_1 > \tilde{\lambda}_2 > \cdots > \tilde{\lambda}_s,$$

and that

$$|a_{jk} - b_{jk}| < \alpha, \quad j,k=1,2,\ldots,p.$$

Then there is a constant C independent of α, such that

(i) $\quad |\delta_k - \lambda_k| < C\alpha, \quad k=1,\ldots,p,$

(ii) $\quad \displaystyle\sum_{k=n_{h-1}+1}^{n_h} \underset{\sim}{u}_k \underset{\sim}{u}_k^* = \sum_{k=n_{h-1}+1}^{n_h} \underset{\sim}{v}_k \underset{\sim}{v}_k^* + G^{(h)}$ with

$$G^{(h)} = (g_{jk}^{(h)}), \quad |g_{jk}^{(h)}| \leq C\alpha, \quad j,k=1,\ldots,p, \quad h=1,\ldots,s.$$

Refer to [15].

LEMMA 3.4. Let $\{X_n, n \geq 1\}$ be a sequence of iid. real random variables such that $EX_1 = 0$ and $EX_1^2 < \infty$. Let $\{a_{nk}\}$ be a double sequence of real numbers such that

$$|a_{nk}| \leq K\, k^{-1/2} \quad \text{for all} \quad k \geq 1,\ n \geq 1,$$

and

$$\sum_k a_{nk}^2 \leq K\, n^{-\alpha} \quad \text{for all } n \geq 1,$$

where $\alpha > 0$ and $K < \infty$ are constants. Then we have

$$\lim_{n\to\infty} \sum_k a_{nk} X_k = 0 \qquad \text{a.s.}$$

Refer to Stout ([13], p.231.)

LEMMA 3.5. Let $g_n(x)$ be a sequence of K-degree polynomials with roots $x_1^{(n)},\dots,x_k^{(n)}$ for each n, and let g(x) be a k-degree polynomials with roots $x_1,\dots,x_k$, $k\le K$. If $g_n(x) \to g(x)$ as $n\to\infty$, then after suitable rearrangement of $x_1^{(n)},\dots,x_K^{(n)}$, we have $x_j^{(n)} \to x_j$, $j=1,2,\dots,k$ and $|x_j^{(n)}| \to \infty$, $j=k+1,\dots,K$.

See Bai ([16]).

THEOREM 3.1. Suppose that in the model (1.1), the conditions (1.2), (1.3), (2.1) and (2.2) are satisfied. Then

$$\lim_{n\to\infty} \hat{q} = q \qquad \text{a.s.}$$

Proof. Assume that $q = \#(\Lambda)$, $\rho = \max_{q<k\le p} |\lambda_k| < 1$, and $\max\{|a_{kj}|, 1\le k\le p, j=1,2,\dots,N\} = K$. Under the model (1.1),

$$\hat{\gamma}^{(r)}_{\ell,m} = \frac{1}{N(n-r)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-r} \overline{Y_j(t+\ell)}\, Y_j(t-m)$$

$$= \frac{1}{N(n-r)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-r} \left(\sum_{k=1}^{p} \bar{\lambda}_k^{t+\ell} \bar{a}_{kj} + \overline{e_j(t+\ell)} \right) \left(\sum_{k=1}^{p} \lambda_k^{t+m} a_{kj} + e_j(t+m) \right), \tag{3.1}$$

where ℓ, $m=0,1,\dots,r$, $r=0,1,\dots,p$. We have

$$\frac{1}{N(n-r)} \sum_{j=1}^{N} \left| \sum_{t=0}^{n-1-r} \sum_{k=q+1}^{p} \bar{\lambda}_k^{t+\ell} \bar{a}_{kj} \right| \left(\left| \sum_{k=1}^{q} \lambda_k^{t+m} a_{kj} \right| + \left| \sum_{k=q+1}^{p} \lambda_k^{t+m} a_{kj} \right| \right)$$

$$\le \frac{K^2}{n-r}(p-q)p \sum_{t=0}^{\infty} \rho^t \le \frac{p^2 K^2}{(n-r)(1-\rho)} = O\left(\frac{1}{n}\right), \quad \ell, m = 0,1,\dots,r. \tag{3.2}$$

By Lemma 3.1, $\sum_{t=0}^{\infty} \rho^t |e_j(t+m)|$ converges a.s., and

$$\frac{1}{N(n-r)}\left|\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\sum_{k=q+1}^{p} \bar{\lambda}_k^{t+\ell} \bar{a}_{kj}\, e_j(t+m)\right|$$

$$\leq \frac{K(p-q)}{N(n-r)} \sum_{j=1}^{N}\sum_{t=0}^{\infty} \rho^t |e_j(t+m)| = 0(\tfrac{1}{n}) \quad \text{a.s.} \tag{3.3}$$

By (3.1) - (3.3), with probability one we have

$$\hat{\gamma}_{\ell m}^{(r)} = \sum_{k=1}^{q} \lambda_k^{m-\ell}\left(\frac{1}{N}\sum_{j=1}^{N}|a_{kj}|^2\right)$$

$$+ \sum_{k,\kappa=1,k\neq\kappa}^{q}\left(\frac{1}{N}\sum_{j=1}^{N} \bar{a}_{j} a_{\kappa j}\, \bar{\lambda}_k^{\ell}\right)\lambda_\kappa^m \frac{1}{n-r}\sum_{t=0}^{n-1-r}(\bar{\lambda}_k \lambda_\kappa)^t$$

$$+ \sum_{k=1}^{q} \bar{\lambda}_k^{\ell-m} \frac{1}{N}\sum_{j=1}^{N} \bar{a}_{kj}\left(\frac{1}{n-r}\sum_{t=0}^{n-1-r} \bar{\lambda}_k^{t+m}\, e_j(t+m)\right)$$

$$+ \sum_{k=1}^{q} \lambda_k^{m-\ell} \frac{1}{N}\sum_{j=1}^{N} a_{kj}\left(\frac{1}{n-r}\sum_{t=0}^{n-1-r} \lambda_k^{t+\ell}\, \overline{e_j(t+\ell)}\right)$$

$$+ \frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r} \overline{e_j(t+\ell)}\, e_j(t+m) + 0\left(\tfrac{1}{n}\right)$$

$$= J_1 + J_{2n} + J_{3n} + J_{4n} + J_{5n} + 0(\tfrac{1}{n}). \tag{3.4}$$

Write $\lambda_k = \exp(i\omega_k)$, $\omega_k \in (0,2\pi)$, $k=1,2,\ldots,q$. Since $\omega_k \neq \omega_\ell$ for $k\neq\ell$,

we have

$$J_{2n} = 0(\tfrac{1}{n}). \tag{3.5}$$

By Lemma 3.2,

$$J_{3n} = 0\left(\sqrt{\frac{\log\log n}{n}}\right) \text{ a.s.,} \quad J_{4n} = 0\left(\sqrt{\frac{\log\log n}{n}}\right) \text{ a.s.} \tag{3.6}$$

By the law of the iterated logarithm of M-dependence sequence,

$$J_{5n} = \begin{cases} O\left(\sqrt{\frac{\log\log n}{n}}\right), & \text{for } \ell \neq m, \\ \sigma^2 + O\left(\sqrt{\frac{\log\log n}{n}}\right), & \text{for } \ell = m, \end{cases} \quad \text{a.s.} \tag{3.7}$$

Put

$$\underset{(r+1)\times q}{\Omega^{(r)}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \bar{\lambda}_1 & \bar{\lambda}_2 & \cdots & \bar{\lambda}_q \\ \bar{\lambda}_1^2 & \bar{\lambda}_2^2 & \cdots & \bar{\lambda}_q^2 \\ \vdots & \vdots & & \vdots \\ \bar{\lambda}_1^r & \bar{\lambda}_2^r & \cdots & \bar{\lambda}_q^r \end{pmatrix}, \quad A = \mathrm{diag}\left[\frac{1}{N}\sum_{j=1}^{N}|a_{1j}|^2, \ldots, \frac{1}{N}\sum_{j=1}^{N}|a_{qj}|^2\right], \quad \Gamma^{(r)} = \sigma^2 I_{r+1} + \Omega^{(r)} A \Omega^{(r)*}. \tag{3.8}$$

Then, by (3.4)-(3.8) we have

$$\hat{\Gamma}^{(r)} = \Gamma^{(r)} + O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \tag{3.9}$$

Let $\hat{\theta}_1^{(r)} \geq \ldots \geq \hat{\theta}_{r+1}^{(r)}$ and $\theta_1^{(r)} \geq \ldots \geq \theta_{r+1}^{(r)}$ be the eigenvalues of $\hat{\Gamma}^{(r)}$ and $\Gamma^{(r)}$ respectively. By Lemma 3.3,

$$\hat{\theta}_k^{(r)} = \theta_k^{(r)} + O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.,} \quad k=1,\ldots,r+1. \tag{3.10}$$

Since rank $(\Omega^{(r)} A \Omega^{(r)*}) = \min(r+1, q)$, we have

$$\begin{aligned} \theta_{r+1}^{(r)} &> \sigma^2 \quad \text{for} \quad r < q. \\ \theta_{r+1}^{(r)} &= \sigma^2 \quad \text{for} \quad r \geq q. \end{aligned} \tag{3.11}$$

Since $Q_r = \hat{\theta}_{r+1}^{(r)}$, we get

$$\lim_{n\to\infty} Q_r = \theta_{r+1}^{(r)} > \sigma^2, \quad \text{a.s., for} \quad r < q, \tag{3.12}$$

and

$$|Q_r - \sigma^2| = O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \quad \text{for} \quad r \geq q. \tag{3.13}$$

From (2.6), (2.7), (3.12) and (3.13), it is easily shown that, with probability one for n large,

$$R_q < R_r \quad \text{for} \quad r \neq q, \quad 1 \leq r \leq p,$$

and, by the definition of $\hat{q}$,

$$\hat{q} = q.$$

The theorem 3.1 is proved.

In the sequel we assume that q is known. For simplicity, we write $\hat{\Gamma}^{(q)} = \hat{\Gamma}$, $\hat{\gamma}_{\ell m}^{(q)} = \hat{\gamma}_{\ell m}$, $\hat{\underset{\sim}{b}}^{(q)} = \hat{\underset{\sim}{b}}$, $\Omega^{(q)} = \Omega$, etc.

THEOREM 3.2. Suppose that in the model (1.1), the conditions (1.2), (2.1) and (2.2) are satisfied. Then, for appropriate ordering, we have

$$\lim_{n\to\infty} \hat{\omega}_k = \omega_k \quad \text{a.s.,} \quad k=1,2,\ldots,q,$$

and

$$\lim_{n\to\infty} Q_q = \sigma^2 \quad \text{a.s.}$$

Proof. Under the conditions of the theorem, (3.2)-(3.5) still hold. By Lemma 3.4,

$$\lim_{n\to\infty} J_{5n} = \begin{cases} 0, & \text{for } \ell \neq m, \\ \sigma^2 & \text{for } \ell = m, \end{cases} \qquad \text{a.s.}$$

It follows that

$$\lim_{n\to\infty} \hat{\Gamma} = \Gamma(= \sigma^2 I_{q+1} + \Omega A \Omega^*) \quad \text{a.s.} \tag{3.14}$$

Define

$$B(z) \triangleq b_q \prod_{k=1}^{q} (z-\lambda_k) \triangleq b_o + b_1 z + \ldots + b_q z^q \tag{3.15}$$

such that $b_q > 0$ and $\sum_{k=o}^{q} |b_k|^2 = 1$. Then $\underset{\sim}{b} = (b_o,\ldots,b_q)' \in B_q$ and

$$\Omega A \Omega^* \underset{\sim}{b} = 0, \quad \Gamma \underset{\sim}{b} = \sigma^2 \underset{\sim}{b}. \tag{3.16}$$

Let $\hat{\theta}_1 \geq \cdots \geq \hat{\theta}_{q+1}$ and $\theta_1 \geq \cdots \geq \theta_{q+1}$ be the eigenvalues of $\hat{\Gamma}$ and Γ respectively. Since rank $(\Omega A \Omega^*) = q$, we have

$$\theta_1 \geq \cdots \geq \theta_q > \theta_{q+1} (=\sigma^2). \tag{3.17}$$

By (3.16) and (3.17), $\underset{\sim}{b}$ is the unit eigenvector of Γ associated with the unique smallest eigenvalue of Γ. Now $\hat{\underset{\sim}{b}} = (\hat{b}_0, \ldots, \hat{b}_q)' \in B_q$ is the unit eigenvector of $\hat{\Gamma}$ associated with its smallest eigenvalue Q_q. Using Lemma 3.3 and (3.14), we get

$$\lim_{n\to\infty} \hat{\underset{\sim}{b}} = \underset{\sim}{b} \quad \text{a.s., and} \quad \lim_{n\to\infty} Q_q = \sigma^2 \quad \text{a.s.} \tag{3.18}$$

By Lemma 3.5, for appropriate ordering, we have

$$\lim_{n\to\infty} \hat{\rho}_k \exp(i\hat{\omega}_k) = \exp(i\omega_k) \quad \text{a.s.,} \quad k=1,2,\ldots,q,$$

which implies that

$$\lim_{n\to\infty} \hat{\rho}_k = 1 \quad \text{a.s. and} \quad \lim_{n\to\infty} \hat{\omega}_k = \omega_k \quad \text{a.s.,} \quad k=1,2,\ldots,q.$$

Theorem 3.2 is proved.

THEOREM 3.3. If (1.2), (2.1) and (2.2) hold, then

$$\lim_{n\to\infty} |a_{kj}|^2 = |a_{kj}|^2 \quad \text{a.s.} \quad \text{for } k=1,\ldots,q, \quad j=1,\ldots,N.$$

Proof. By the theorem 3.2,

$$\lim_{n\to\infty} \hat{\lambda}_k = \lambda_k \quad \text{a.s., } k=1,\ldots,q.$$

Define

$$M = (\mu_{\ell m}) = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_q \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \lambda_1^{q-1} & \lambda_2^{q-1} & \lambda_q^{q-1} \end{pmatrix}^{-1}.$$

Then we have

$$\hat{\mu}_{\ell m} \to \mu_{\ell m} \quad \text{a.s.,} \quad \ell, m = 1, \ldots, q, \tag{3.19}$$

which implies that

$$\lim_{n\to\infty} \hat{\mu}_{k\ell} \lambda_m^{\ell-1} = \delta_{km} = \begin{cases} 1, & k=m, \\ 0, & k \neq m, \end{cases} \quad \text{a.s.} \tag{3.20}$$

By (1.1), (3.19) and (3.20), for $k=1,\ldots,q$ we have

$$\frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big| \sum_{\ell=1}^{q} \hat{\mu}_{k\ell} Y_j(t+\ell-1) \Big|^2$$

$$= \frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big| \sum_{\ell=1}^{q} \hat{\mu}_{k\ell} \Big(\sum_{m=1}^{p} a_{mj} \lambda_m^{t+\ell-1} + e_j(t+\ell-1) \Big) \Big|^2$$

$$= \frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big| \sum_{m=1}^{p} \delta_{km} a_{mj} \lambda_m^t + \sum_{\ell=1}^{q} \sum_{m=1}^{p} |a_{mj}| \cdot o(1)$$

$$+ \sum_{\ell=1}^{q} \mu_{k\ell} e_j(t+\ell-1) + \sum_{\ell=1}^{q} \big| e_j(t+\ell-1) \big| \cdot o(1) \Big|^2 \quad \text{a.s.}$$

$$\overset{\text{a.s.}}{=} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big| a_{kj} \lambda_k^t + O(1) + \sum_{\ell=1}^{q} \mu_{k\ell} e_j(t+\ell-1) + \sum_{\ell=1}^{q} \big| e_j(t+\ell-1) \big| \cdot o(1) \Big|^2$$

$$\overset{\text{a.s.}}{=} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big\{ |a_{kj}|^2 + \sum_{\ell,m=1}^{q} \mu_{k\ell} \bar{\mu}_{km} e_j(t+\ell-1) \overline{e_j(t+m-1)} + o(1)$$

$$+ \sum_{\ell=1}^{q} \Big(|e_j(t+\ell-1)|^2 + |e_j(t+\ell-1)| \Big) \cdot o(1) + a_{kj} \lambda_k^t \sum_{\ell=1}^{q} \mu_{k\ell} e_j(t+\ell-$$

$$+ \bar{a}_{kj}\bar{\lambda}_k^t \sum_{\ell=1}^{q} \bar{\mu}_{k\ell}\overline{e_j(t+\ell-1)}\Big\}. \tag{3.21}$$

By the SLLN,

$$\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{\ell,m=1}^{q} \mu_{k\ell}\bar{\mu}_{km}e_j(t+\ell-1)\overline{e_j(t+m-1)}$$

$$= \sum_{\ell,m=1}^{q} \mu_{k\ell}\bar{\mu}_{km}\delta_{\ell m}\sigma^2 = \sum_{\ell=1}^{q} |\mu_{k\ell}|^2\sigma^2 \quad \text{a.s.} \tag{3.22}$$

$$\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q}\Big(|e_j(t+\ell-1)|^2 + |e_j(t+\ell-1)|\Big) = \sigma^2 + E|e_1(o)|, \text{ a.s.} \tag{3.23}$$

By Lemma 3.4,

$$\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} a_{kj}\lambda_k^t \sum_{\ell=1}^{q} \mu_{k\ell}e_j(t+\ell-1)$$

$$= \sum_{\ell=1}^{q} a_{kj}\mu_{k\ell} \lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \lambda_k^t e_j(t+\ell-1) = 0 \quad \text{a.s.} \tag{3.24}$$

By (3.21)-(3.24),

$$\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big|\sum_{\ell=1}^{q} \hat{\mu}_{k\ell}Y_j(t+\ell-1)\Big|^2 = |a_{kj}|^2 + \sum_{\ell=1}^{q} |\mu_{k\ell}|^2\sigma^2, \text{ a.s.} \tag{3.25}$$

From (2.11), (3.18), (3.19) and (3.25), the theorem 3.3 follows.

Remark 3.1. If $q < p$ and we estimate $\lambda_1,\ldots,\lambda_q$ using (1.5)-(1.8) directly, then we have

$$\lim_{n\to\infty} \hat{\Gamma}^{(p)} = \Gamma^{(p)} \quad \text{a.s.}$$

and $\theta_1^{(p)} \geq \ldots \geq \theta_q^{(p)} > \theta_{q+1}^{(p)} = \ldots = \theta_{p+1}^{(p)}$ $(=\sigma^2)$ are the eigenvalues of $\Gamma^{(p)}$. All eigenvectors of $\Gamma^{(p)}$ associated with σ^2 consist of a (p-q) dimension subspace. Assume that $\hat{\underset{\sim}{b}}^{(p)} \in B_p$ such that

$$Q_p(\hat{\underset{\sim}{b}}^{(p)}) = \min\{Q_p(\underset{\sim}{b}^{(p)}) : \underset{\sim}{b}^{(p)} \in B_p\},$$

then $\underset{\sim}{\hat{b}}^{(p)}$ is the eigenvector of $\hat{\Gamma}^{(p)}$ associated with the smallest eigenvalue of $\hat{\Gamma}^{(p)}$. We do not know whether $\underset{\sim}{\hat{b}}^{(p)}$ converges. In general, we do not know whether there are q roots among all roots of $\sum_{k=0}^{p} \hat{b}_k^{(p)} z^k$ which tend to $\{\lambda_k,\ k=1,2,\ldots,q\}$.

Remark 3.2. Suppose that in the model (2.12), the condition (1.2) hold. Then $a_{0j} = \sum_{t=0}^{n-1} Y_j(t)/n$ is a strongly consistent estimate of a_{0j}. For those procedures of detection and estimation discussed in Remark 2.1, Theorem 3.1 - 3.3 are also true.

Finally, we establish the asymptotic normality of Q_q and $(\hat{\omega}_k,\ k=1,2,\ldots,q)$. To this end, we assume that under the model (1.1), the conditions (2.1), (2.2) are satisfied, and $e_j(t)$, $j=1,2,\ldots,N$, $t=0,1,\ldots,n-1$, are iid. complex variables, $e_j(t) = e_{j1}(t) + ie_{j2}(t)$ and $e_{j2}(t)$ are both real numbers, which satisfy the following conditions.

$$\begin{gathered} Ee_j(t) = 0 \quad Ee_{j1}(t)^2 = Ee_{j2}(t)^2 = 1/2\ \sigma^2, \\ Ee_{j1}(t)e_{j2}(t) = 0 \ \text{ and } \ \mathrm{Var}(|e_j(t)|^2) = \alpha\sigma^4 \quad \text{with } \alpha>0. \end{gathered} \tag{3.26}$$

Put

$$U = \begin{pmatrix} u_0 & \bar{u}_1 & \cdot & \cdots & \bar{u}_q \\ u_1 & u_0 & \cdot & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \bar{u}_1 \\ u_q & \cdots & \cdots & u_1 & u_0 \end{pmatrix}$$

where $u_0, u_1, \ldots, u_q$ are independent, and

(i) $u_0 \sim N_r(0, \alpha\sigma^4)$,

(ii) $u_k \sim N_c(o,\sigma^4)$, $K=1,2,\ldots,q.$ (3.28)

Here N_c, N_r denote complex and real normal distribution respectively.

Define $\underset{\sim}{b} = (b_0,b_1,\ldots,b_q)'$ by (3.15). Put

$$A = \text{diag}\,[\frac{1}{N}\sum_{j=1}^{N}|a_{1j}|^2,\ldots,\frac{1}{N}\sum_{j=1}^{N}|a_{qj}|^2],$$

$$\underset{(q+1)\times q}{\Omega} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \bar{\lambda}_1 & \bar{\lambda}_2 & \ldots & \bar{\lambda}_q \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \bar{\lambda}_1^q & \bar{\lambda}_2^q & \ldots & \bar{\lambda}_q^q \end{pmatrix},$$

$$\zeta_n = \sqrt{N(n-q)}(Q_q-\sigma^2), \quad T_{nk} = \sqrt{N(n-q)}(\hat{\rho}_k-1), \tag{3.29}$$

$$\Delta_{nk} = \sqrt{N(n-q)}(\hat{\omega}_k-\omega_k), \quad k=1,2,\ldots,q,$$

$$\underset{\sim}{T}_n = (T_{n1},\ldots,T_{nq})' \quad \text{and} \quad \underset{\sim}{\Delta}_n = (\Delta_{n1},\ldots,\Delta_{nq})'.$$

Write

$$B(z) = \sum_{k=o}^{q} b_k z^k,$$

$$D(\exp(i\omega)) = -i\frac{d}{d\omega}B(e^{i\omega})$$

and

$$G = \text{diag}[D(\exp(i\omega_1)),\ldots,D(\exp(i\omega_q))]. \tag{3.30}$$

We have the following

THEOREM 3.4. Suppose that in the model (1.1), the conditions (2.1), (2.2) and (3.26) are satisfied. Then we have

$$\zeta_n \xrightarrow{D} = \underset{\sim}{b}^* \, U \, \underset{\sim}{b}, \tag{3.31}$$

$$\underset{\sim}{T}_n + i\underset{\sim}{\Delta}_n \xrightarrow{D} G^{-1}A^{-1}(\Omega^*\Omega)^{-1}\Omega^* \, U \, \underset{\sim}{b}, \tag{3.32}$$

as $n \to \infty$.

Here we quote the following

LEMMA 3.6. Suppose that the condition (3.26) holds. Then

$$\frac{1}{\sqrt{n-q}} \sum_{t=0}^{n-1-q} \lambda_k^{t+\ell} \overline{\ell_j(t+\ell)} \xrightarrow{D} v_{kj},$$

$$k=1,2,\ldots,q, \quad j=1,2,\ldots,N, \quad \ell=0,1,\ldots,q,$$

$$\frac{1}{n-q} \sum_{t=0}^{n-1-q} (|e_j(t+\ell)|^2 - \sigma^2) \xrightarrow{D} u_{0j}, \quad j=1,\ldots,N, \quad \ell=0,1,\ldots,q,$$

$$\frac{1}{n-q} \sum_{t=0}^{n-1-q} \overline{e_j(t+\ell)} e_j(t+m) \xrightarrow{D} u_{\ell-m,j}, \quad j=1,\ldots,N, \quad 0 \le m < \ell \le q.$$

Here u_{kj}'s and v_{kj}'s are independent of each other, and

(i) $u_{0j} \sim N_r(0, \alpha\sigma^4)$, $\quad j=1,\ldots,N$.

(ii) $u_{kj} \sim N_c(0,\sigma^4)$, $\quad j=1,\ldots,N, \quad k=1,\ldots,q$. (3.33)

(iii) $v_{kj} \sim N_c(0,\sigma^2)$, $\quad j=1,\ldots,N, \quad k=1,\ldots,q$.

Refer to Lemma 4.1 in [9].

The proof of Theorem 3.4 runs along the line as in the proof of Theorem 4.1 in [9], so the details are omitted.

Remark 3.3. For the model (2.12), Theorem 3.4 applies those estimates discussed in Remark 2.1.

4. NON-EXISTENCE OF CONSISTENT ESTIMATES OF λ_k, $k \in \Lambda^c$ AND $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_N$ WHEN $\Lambda^c \neq \phi$

Throughout this section, N is fixed and $n \to \infty$. For non-existence of consistent estimate of λ_k, $k \in \Lambda^c$, we have the following

THEOREM 4.1. Suppose that in the model (1.1), $e_j(t)$, $j=1,\ldots,N$, $t=0,1,\ldots,n-1$, are iid., $e_j(t) \sim N_c(0,\sigma^2)$ with $0 < \sigma^2 < \infty$, and the parameter space of $\underset{\sim}{\lambda} = (\lambda_1,\ldots,\lambda_p)'$ contains two points $\underset{\sim}{\lambda}^{(k)} = (\lambda_1,\ldots,\lambda_{p-1}, \lambda_p^{(k)})'$, k=1,2 such that $\underset{\sim}{\lambda}^{(1)} \neq \underset{\sim}{\lambda}^{(2)}$, $|\lambda_p^{(k)}| < 1$, k=1,2, and $\sum_{j=1}^{N} |a_{pj}|^2 > 0$. Then no consistent estimate of λ_p exists as is N fixed and $n \to \infty$.

Proof. It suffices to show that a consistent estimate of λ_p cannot exist even when $\{a_{kj}\}$ and σ^2 are known. Hence, without loss of generality, we assume $\sigma^2 = 2$.

Introduce the prior distribution H:

$$H(\underset{\sim}{\lambda}^{(1)}) = H(\underset{\sim}{\lambda}^{(2)}) = 1/2$$

and the square loss $|d-\lambda_p|^2$. Write

$$f_k = (2\pi)^{-nN} \exp\{-\tfrac{1}{2} \sum_{j=1}^{N} \sum_{t=0}^{n-1} |Y_j(t) - \sum_{\ell=1}^{p-1} \lambda_\ell^t a_{\ell j} - (\lambda_p^{(k)})^t a_{pj}|^2\}, \quad k = 1,2.$$

Under the above prior distribution and loss function, the Bayesian estimate of λ_p is

$$\tilde{\lambda}_p = (f_1 \lambda_p^{(1)} + f_2 \lambda_p^{(2)})/(f_1+f_2).$$

Denote by $R(\tilde{\lambda}_p)$ the Bayesian risk of $\tilde{\lambda}_p$, we have

$$R(\tilde{\lambda}_p) \geq \tfrac{1}{2} E(|\tilde{\lambda}_p - \lambda_p^{(1)}|^2 | k = 1)$$

$$= \tfrac{1}{2}E\left\{\left(\frac{f_2}{f_1+f_2}\right)^2 \middle| k = 1\right\}|\lambda_p^{(1)}-\lambda_p^{(2)}|^2. \qquad (4.1)$$

Noticing that $Y_j(t) - \sum_{\ell=1}^{p-1} \lambda_\ell^t a_{\ell j} - (\lambda_p^{(1)})^t a_{pj} = e_j(t)$ when $k = 1$, we have

$$\log \frac{f_2}{f_1} \geq -\tfrac{1}{2}\sum_{j=1}^{N} \sum_{t=0}^{n-1} |a_{pj}|^2 |(\lambda_p^{(1)})^t - (\lambda_p^{(2)})^t|^2$$

$$- |\sum_{j=1}^{N} \sum_{t=0}^{n-1} ((\lambda_p^{(2)})^t - (\lambda_p^{(1)})^t) a_{pj} e_j(t)|. \qquad (4.2)$$

Since $|\lambda_p^{(1)}| < 1$, $|\lambda_p^{(2)}| < 1$, we have

$$\lim_{n\to\infty} \sum_{j=1}^{N} \sum_{t=0}^{n-1} |a_{pj}|^2 |(\lambda_p^{(1)})^t - (\lambda_p^{(2)})^t|^2 < \infty. \qquad (4.3)$$

Also, by Lemma 3.1, the second term of the right hand side of (4.2) converges with probability one to a finite random variable. From this, (4.2) and (4.3), it follows that there exists a positive constant K such that

$$P(f_2/f_1 > K|k = 1) \geq 1/2$$

for n sufficiently large. Hence, form (4.1) we obtain

$$R(\tilde{\lambda}_p) \geq \frac{1}{4}\left(\frac{K}{1+K}\right)^2 |\lambda_p^{(1)}-\lambda_p^{(2)}|^2 > 0 \qquad (4.4)$$

for n large.

But if ξ_n is a consistent estimate of λ_p, then define

$$\tilde{\xi}_n = \begin{cases} \lambda_p^{(1)}, & \text{if } |\xi_n-\lambda_p^{(1)}| \leq |\xi_n-\lambda_p^{(2)}| \\ \lambda_p^{(2)}, & \text{otherwise.} \end{cases}$$

We shall have

$$\tilde{\xi}_n \xrightarrow{P} \lambda_p^{(k)} \quad \text{for} \quad \lambda = \lambda^{(k)}, \quad k = 1,2.$$

Since $\tilde{\xi}_n$ is bounded, by the dominated convergence theorem, we have

$$R(\tilde{\xi}_n) = 1/2\ E(|\tilde{\xi}_n-\lambda_p^{(1)}|^2|\underset{\sim}{\lambda} = \underset{\sim}{\lambda}^{(1)}) + 1/2\ E(|\tilde{\xi}_n-\lambda_p^{(2)}|^2|\underset{\sim}{\lambda} = \underset{\sim}{\lambda}^{(2)}) \to 0 \quad (4.5)$$

as $n \to \infty$, where $R(\tilde{\xi}_n)$ is the Bayesian risk of $\tilde{\xi}_n$. But this contradicts (4.4) in view of the fact that $\tilde{\lambda}_p$ is the Bayesian estimate of λ_p, and the theorem is proved.

For the existence problem of $a_j = (a_{ij},\ldots,a_{pj})'$ when $\Lambda^c \neq \phi$, we have the following

THEOREM 4.2. Suppose that in the model (1.1), $e_j(t)$, $j = 1,\ldots,N$, $t = 0,1,\ldots,n-1$, are iid., $e_j(t) \sim N_c(0,\sigma^2)$ with $0 < \sigma^2 < \infty$. Also, some component of $\underset{\sim}{\lambda}$, say λ_p, has a module less than one, and $\lambda_k \neq \lambda_\ell$ if $k \neq \ell$. Then no consistent estimates can be found for $a_{p1},\ldots,a_{pN}$.

Proof. As in the proof of Theorem 4.1, we may assume that $\lambda_1,\ldots,\lambda_p$ and a_{kj}, $k = 1,\ldots,p-1$, $j = 1,\ldots,N$, are known. Also without loss of generality we may assume $N = 1$. Write $Y_1(t) - \sum_{k=1}^{p-1} a_{k1}\lambda_k^t = X(t)$, and for simplicity, write $\lambda_p = \lambda$, $a_p = \beta$. Then the model (1.1) is reduced as the following linear model:

$$X(t) = \beta\lambda^t + e(t), \quad t = 0,1,\ldots,n-1, \qquad (4.6)$$

where λ is known and $|\lambda| < 1$. It is desired to show that there is no consistent estimate for β.

Let $\hat{\beta}$ denote the LS estimate of β. By a theorem of Drygas [17], the consistency of $\hat{\beta}$ is equivalent to $\mathrm{Var}(\hat{\beta}) \to 0$. But from

$$\hat{\beta} = \left(\sum_{t=0}^{n-1}|\lambda|^{2t}\right)^{-1} \sum_{t=0}^{n-1}\bar{\lambda}^t X(t),$$

and

$$\mathrm{Var}(\hat{\beta}) = \sigma^2\left(\sum_{t=0}^{n-1}|\lambda|^{2t}\right)^{-1} \to \sigma^2(1-|\lambda|^2) \neq 0,$$

we know that $\hat{\beta}$ is not consistent.

Since $\{e(t)\}$ is a sequence of iid. variables with a common normal distribution, it follows by a theorem of Ker-Chan Li [18] that there cannot exist any consistent estimate for β. The theorem 4.2 is proved.

5. THE CASE WHERE n IS FIXED AND $N \to \infty$.

In this section, we assume that $n \geq p+1$ is fixed and N tends to infinity. Consider the model (1.1). Assume that $\lambda_k \neq \lambda_\ell$ if $k \neq \ell$ (note that the condition $|\lambda_k| = 1$ can be dropped), and that (1.2) is true. We can use the EVLP method described in section 1 to obtain estimates $\hat{\lambda}_k's$ of $\lambda_k's$ (refer to (1.5)-(1.8) and so on). We have the following

THEOREM 5.1. Suppose that under the model (1.1), $\lambda_k \neq \lambda_\ell$ if $k \neq \ell$, and (1.2) holds. Also, $n \geq p+1$ is fixed and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \bar{\underset{\sim}{a}}_j \underset{\sim}{a}_j' = \Psi$$

exists, where $\underset{\sim}{a}_j' = (a_{1j}, a_{2j}, \ldots, a_{pj})$, $\Psi = (\psi_{\ell m})_1^p$ is a $p \times p$ positive definite matrix. Then, $\hat{\lambda}_1, \ldots, \hat{\lambda}_p$ and Q_p are strongly consistent estimates of $\lambda_1, \ldots, \lambda_p$ and σ^2.

Proof. By (1.5)-(1.7), Q_p is the smallest eigenvalue of the matrix $\hat{\Gamma} = (\hat{\gamma}_{\ell m})$, $\ell, m = 0, 1, \ldots, p$, and $\hat{\underset{\sim}{b}}$ is the corresponding eigenvector, where

$$\hat{\gamma}_{\ell m} = \frac{1}{N(n-p)} \sum_{j=1}^{N} \sum_{t=0}^{n-1-p} \overline{Y_j(t+\ell)}\, Y_j(t+m), \quad \ell, m = 0, 1, \ldots, p. \qquad (5.1)$$

By (1.1),

$$\hat{\gamma}_{\ell m} = \sum_{k,\kappa=1}^{p} \frac{1}{n-p} \sum_{t=0}^{n-1-p} \bar{\lambda}_k^{t+\ell} \lambda_\kappa^{t+m} \frac{1}{N} \sum_{j=1}^{N} \bar{a}_{kj} a_{\kappa j}$$

$$+ \sum_{k=1}^{p} \frac{1}{n-p} \sum_{t=0}^{n-1-p} \bar{\lambda}_k^{t+\ell} \frac{1}{N} \sum_{j=1}^{N} \bar{a}_{kj}\, e_j(t+m)$$

$$+ \sum_{k=1}^{p} \frac{1}{n-p} \sum_{t=0}^{n-1-p} \lambda_k^{t+m} \frac{1}{N} \sum_{j=1}^{N} a_{kj}\, \overline{e_j(t+\ell)}$$

$$+\frac{1}{N(n-p)}\sum_{t=0}^{n-1-p}\sum_{j=1}^{N}\overline{e_j(t+\ell)}\, e_j(t+m)$$

$$\triangleq I_{1N}+I_{2N}+I_{3N}+I_{4N}. \tag{5.2}$$

We have

$$\lim_{N\to\infty} I_{1N}=\sum_{k,\kappa=1}^{p}\lambda_k^{\ell}\lambda_\kappa^{m}\frac{1}{n-p}\sum_{t=0}^{n-1-p}\lambda_k^{t}\,\psi_{k\kappa}\lambda_\kappa^{t},\quad \ell,m=0,1,\ldots,p. \tag{5.3}$$

By Lemma 3.4,

$$\lim_{N\to\infty} I_{2N}=\lim_{N\to\infty} I_{3N}=0 \quad \text{a.s.} \tag{5.4}$$

By the SLLN,

$$\lim_{N\to\infty} I_{4N}=\sigma^2\delta_{\ell m},\ \text{a.s.},\quad \ell,m=0,1,\ldots,p. \tag{5.5}$$

Write

$$\underset{(p+1)\times p}{\Omega}=\begin{pmatrix} 1 & 1 & \cdots & 1\\ \bar\lambda_1 & \bar\lambda_2 & \cdots & \bar\lambda_p\\ \bar\lambda_1^2 & \bar\lambda_2^2 & & \bar\lambda_p^2\\ \vdots & \vdots & & \vdots\\ \bar\lambda_1^p & \bar\lambda_2^p & \cdots & \bar\lambda_p^p\end{pmatrix},\qquad \Omega_1=\mathrm{diag}[\lambda_1,\ldots,\lambda_p]. \tag{5.6}$$

By (5.2) - (5.5), we have

$$\lim_{N\to\infty}\hat\Gamma=\Gamma \quad \text{a.s.} \tag{5.7}$$

where

$$\Gamma=\sigma^2 I_{p+1}+\frac{1}{n-p}\sum_{t=0}^{n-1-p}\Omega\bar\Omega_1^{t}\Psi\Omega_1^{t}\Omega^{*}. \tag{5.8}$$

Noticing that rank $\left(\sum_{t=0}^{n-1-p}\Omega\bar\Omega_1^{t}\Psi\Omega_1^{t}\Omega^{*}\right)=p$, we can finish the proof by repeating the argument used in the proof of Theorem 3.2.

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